Vulnerable Option

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Agenda

- Introduction
  - Vulnerable Option
  - Review
- The Model
  - Assumption
- Valuation Method
- Result
- Conclusion

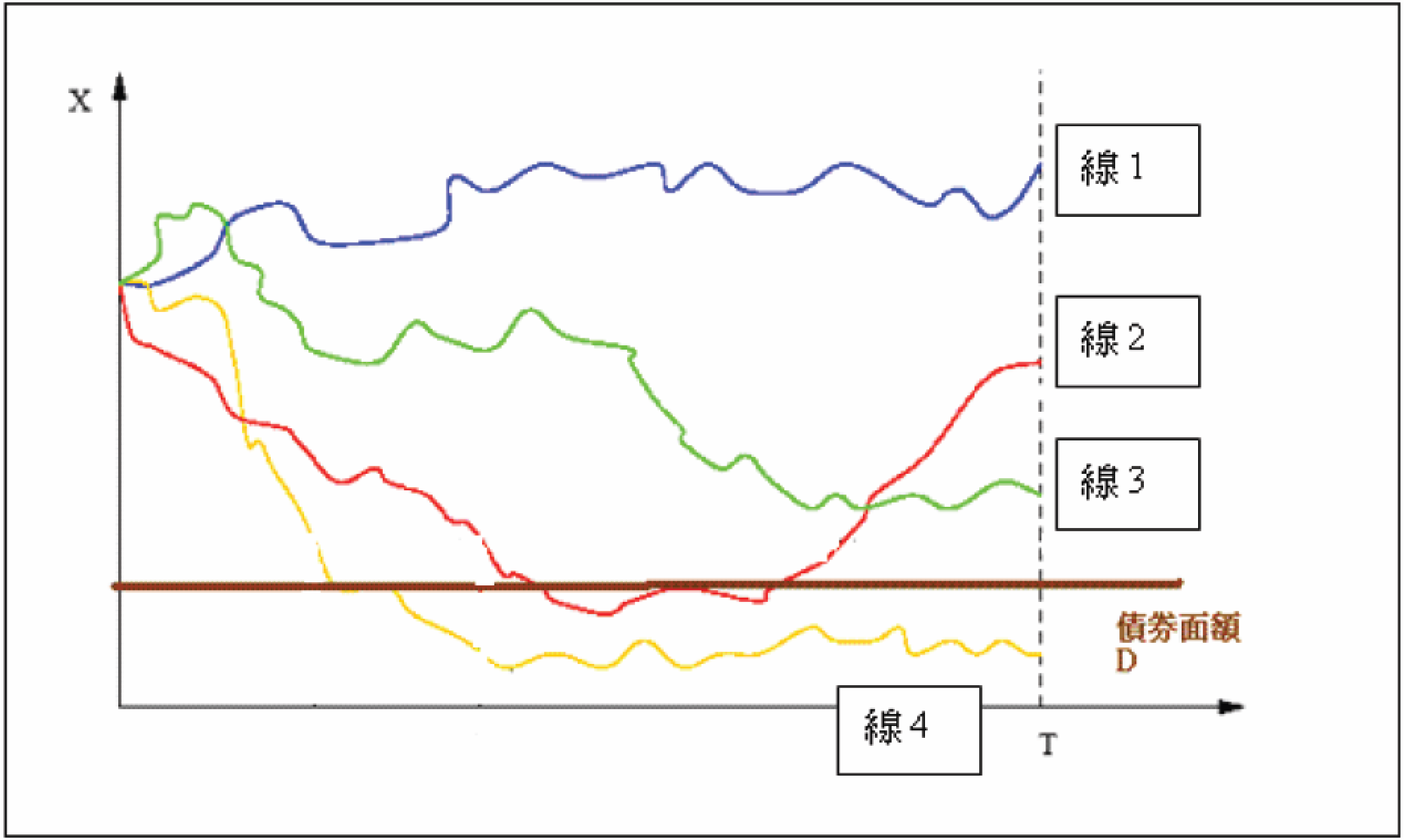
Introduction
Vulnerable Option

- Credit risk
  - Reference entity risk
  - Counterparty risk
    - Vulnerable Option

- Example:
  - 雷曼兄弟賣一個標的資產為希臘國債的衍生性金融商品。

Introduction - FPM

- Merton’s Model & First Passage Model
  - Merton’s Model :
    - Firm asset = option (Equity) + zero coupon bond
  - First Passage Model (FPM)
    - A default boundary regards with time.
    - Can use the barrier option valuation method to valuate.
- Vulnerable Option apply FPM model
  - Let the “nominal claim” be an option of the reference option
    - use the barrier option valuation method to valuate.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Review

- Johnson and Stulz (1987)

$$c = e^{-r(T-t)}\left[E^{*}\left\{ \begin{array}{cc} S_T - K & S_T > K, V_T > S_T - K \\ V_T & if\ S_T > K, V_T \le S_T - K \\ 0 & otherwise \end{array} \right\} \right]$$

- Klein (1996)

$$c = e^{-r(T-t)}\left[E^{*}\left\{ \begin{array}{cc} S_T - K & S_T > K, V_T > D* \\ (1-\alpha)V_T \dfrac{S_T - K}{D*} & if\ S_T > K, V_T \le D* \\ 0 & otherwise \end{array} \right\} \right]$$

Review

- Klein and Inglis (2001)

$$c = e^{-r(T-t)}\left[E^{*}\left\{ \begin{array}{cc} S_T - K & S_T > K, V_T > D^{*} + S_T - K \\ (1-\alpha)V_T \dfrac{S_T - K}{D^{*} + S_T - K} & \text{if } S_T > K, V_T \leq D^{*} + S_T - K \\ 0 & \text{otherwise} \end{array} \right\} \right]$$

- Our Valuation Equation

$$c = e^{-r(T-t)}\left[E^{*}\left\{ \begin{array}{ccc} S_T - K & & \min\limits_{0 \leq \tau \leq T} S(\tau) > B(\tau), S_T > K, V_T > D^{*} + S_T - K \\ (1-\alpha)V_t \dfrac{S_T - K}{D^{*} + S_T - K} & \text{if} & \min\limits_{0 \leq \tau \leq T} S(\tau) > B(\tau), S_T > K, V_T \leq D^{*} + S_T - K \\ 0 & & \text{otherwise} \end{array} \right\} \right]$$

With FPM: $S_T - K \Rightarrow c(t)$ c(t) is the value of reference entity

Assumption

- **Assumption 1**: Firm asset V $\frac{dV}{V} = \mu_V dt + \sigma_V dZ_V$
- **Assumption 2**: Reference entity $\frac{dS}{S} = \mu_S dt + \sigma_S dZ_S$
  - Correlation Z_V and Z_S : ρ_{VS}
- **Assumption 3**: Market is "perfect" and "frictionless," no transaction cost or tax, and trade in continuous time.
- **Assumption 4**: Default occurs when writer's asset V_t equal or less in a threshold $e^{-\gamma(T-t)} D^* + c_t$

Assumption

- **Assumption 5**: The nominal claim of option holder is the value of reference option at time t.
- **Assumption 6**: Upon resolution of the financial distress, the option holder receives (1-w)*nominal claim, where *w* represents the percentage write-down of the nominal claim and can be implied by bankruptcy cost.
- **Assumption 7**: The percentage write-down of the nominal claim *w* is $w = 1-(1-\alpha)V_t/(e^{-\gamma(T-t)}D^* + c_t)$

Valuation Method

- Modify the tree model in 陳博宇(2009).
- Tree construct summary:
  - Orthogonalize two random variables, and output new random variables X and Y with no correlation.
  - Expand one variables in tree structure (X in this thesis,) and compute the barrier in another variable for each node in X tree.
  - Forward construct whole tree using BTT.
  - Backward induction.

Orthogonalization (1)

$$\frac{dV}{V} = rdt + \sigma_V dZ_V \Rightarrow d\ln V(t) = \left(r - \frac{\sigma_V^{\ 2}}{2}\right)dt + \sigma_V dZ_V$$

correlation ρ

$$\frac{dS}{S} = rdt + \sigma_S dZ_S \Rightarrow d\ln S(t) = \left(r - \frac{\sigma_S^{\ 2}}{2}\right)dt + \sigma_S dZ_S$$

Let $dZ \perp dZ_S$ ➔ $dZ_V = \rho dZ_S + \sqrt{1-\rho^2}dZ$

$$d\ln V(t) = \left(r - \frac{\sigma_V^{\ 2}}{2}\right)dt + \sigma_V\left(\rho dZ_S + \sqrt{1-\rho^2}dZ\right)$$

Orthogonalization (2)

$$\begin{bmatrix} d\ln S(t) \\ d\ln V(t) \end{bmatrix} = \begin{bmatrix} r - \frac{\sigma_S{}^2}{2} \\ r - \frac{\sigma_V{}^2}{2} \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 \\ \sigma_V \rho & \sigma_V \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dZ_S \\ dZ \end{bmatrix}$$

$$\begin{bmatrix} \sigma_S & 0 \\ \sigma_V \rho & \sigma_V \sqrt{1-\rho^2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sigma_S} & 0 \\ \frac{-\rho}{\sigma_S \sqrt{1-\rho^2}} & \frac{1}{\sigma_V \sqrt{1-\rho^2}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sigma_S} & 0 \\ \frac{-\rho}{\sigma_S \sqrt{1-\rho^2}} & \frac{1}{\sigma_V \sqrt{1-\rho^2}} \end{bmatrix} \begin{bmatrix} d\ln S(t) \\ d\ln V(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_S} & 0 \\ \frac{-\rho}{\sigma_S \sqrt{1-\rho^2}} & \frac{1}{\sigma_V \sqrt{1-\rho^2}} \end{bmatrix} \begin{bmatrix} r - \frac{\sigma_S{}^2}{2} \\ r - \frac{\sigma_V{}^2}{2} \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dZ_S \\ dZ \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} dX(t) \\ dY(t) \end{bmatrix} \equiv \begin{bmatrix} \frac{1}{\sigma_S} d\ln S(t) \\ \frac{d\ln V(t)}{\sigma_V \sqrt{1-\rho^2}} - \frac{\rho d\ln S(t)}{\sigma_S \sqrt{1-\rho^2}} \end{bmatrix} = \begin{bmatrix} \frac{r - \frac{\sigma_S{}^2}{2}}{\sigma_S} \\ \frac{\left(r - \frac{\sigma_V{}^2}{2}\right)}{\sigma_V \sqrt{1-\rho^2}} - \frac{\rho\left(r - \frac{\sigma_S{}^2}{2}\right)}{\sigma_S \sqrt{1-\rho^2}} \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dZ_S \\ dZ \end{bmatrix}$$

Orthogonalization (3)

$$dX(t) = \frac{1}{\sigma_S}\left(r - \frac{\sigma_S{}^2}{2}\right)dt + dZ_S$$

$$dY(t) = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{1}{\sigma_V}\left(r - \frac{\sigma_V{}^2}{2}\right) - \frac{1}{\sigma_S}\rho\left(r - \frac{\sigma_S{}^2}{2}\right)\right)dt + dZ$$

$$Let\ X(0) \equiv 0, Y(0) \equiv 0$$

$$X(t) - X(0) = X(t) = \int_0^t dX(t) = \int_0^t \tfrac{1}{\sigma_S} d\ln S(t) = \frac{1}{\sigma_S}\left[\ln S(t) - \ln S(0)\right] = \frac{1}{\sigma_S}\ln\frac{S(t)}{S(0)}$$

$$Y(t) - Y(0) = Y(t) = \int_0^t dY(t) = \int_0^t \frac{d\ln V(t)}{\sigma_V\sqrt{1-\rho^2}} - \frac{\rho d\ln S(t)}{\sigma_S\sqrt{1-\rho^2}}$$

$$= \frac{1}{\sigma_V\sqrt{1-\rho^2}}\int_0^t d\ln V(t) - \frac{\rho}{\sigma_S\sqrt{1-\rho^2}}\int_0^t d\ln S(t) = \frac{1}{\sigma_V\sqrt{1-\rho^2}}\ln\frac{V(t)}{V(0)} - \frac{\rho}{\sigma_S\sqrt{1-\rho^2}}\ln\frac{S(t)}{S(0)}$$

$$= \frac{1}{\sqrt{1-\rho^2}}\left[\frac{1}{\sigma_V}\ln\frac{V(t)}{V(0)} - \frac{\rho}{\sigma_S}\ln\frac{S(t)}{S(0)}\right]$$

Orthogonalization (4)

$$X(t) = \frac{1}{\sigma_S} \ln \frac{S(t)}{S(0)} \Rightarrow S(t) = S(0) e^{\sigma_S X(t)}$$

$$Y(t) = \frac{1}{\sqrt{1-\rho^2}} \left[\frac{1}{\sigma_V} \ln \frac{V(t)}{V(0)} - \frac{\rho}{\sigma_S} \ln \frac{S(t)}{S(0)} \right] = \frac{1}{\sqrt{1-\rho^2}} \left[\frac{1}{\sigma_V} \ln \frac{V(t)}{V(0)} - \rho X(t) \right]$$

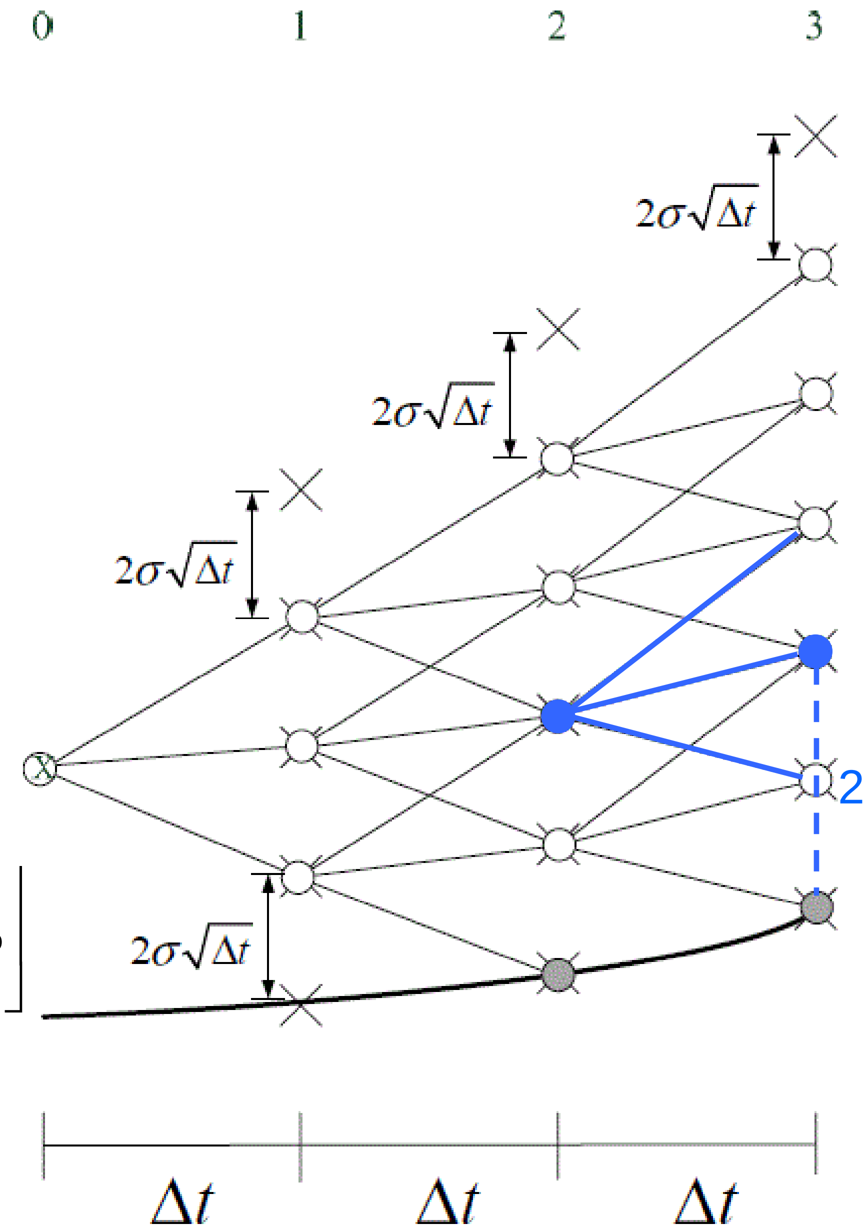
$$\Rightarrow V(t) = V(0) \exp\left\{ \sigma_V \left[\sqrt{1-\rho^2}\, Y(t) + \rho X(t) \right] \right\}$$

Expand X tree

- We use stair tree to expand X variable
- Use the equation to find the position of next time.

$$X_{middle}(t+\Delta t) = \left\lfloor \frac{X(t) + \mu\Delta t - B_X(t+\Delta t)}{2\sigma_X\sqrt{\Delta t}} + 0.5 \right\rfloor$$

- Compute option value.

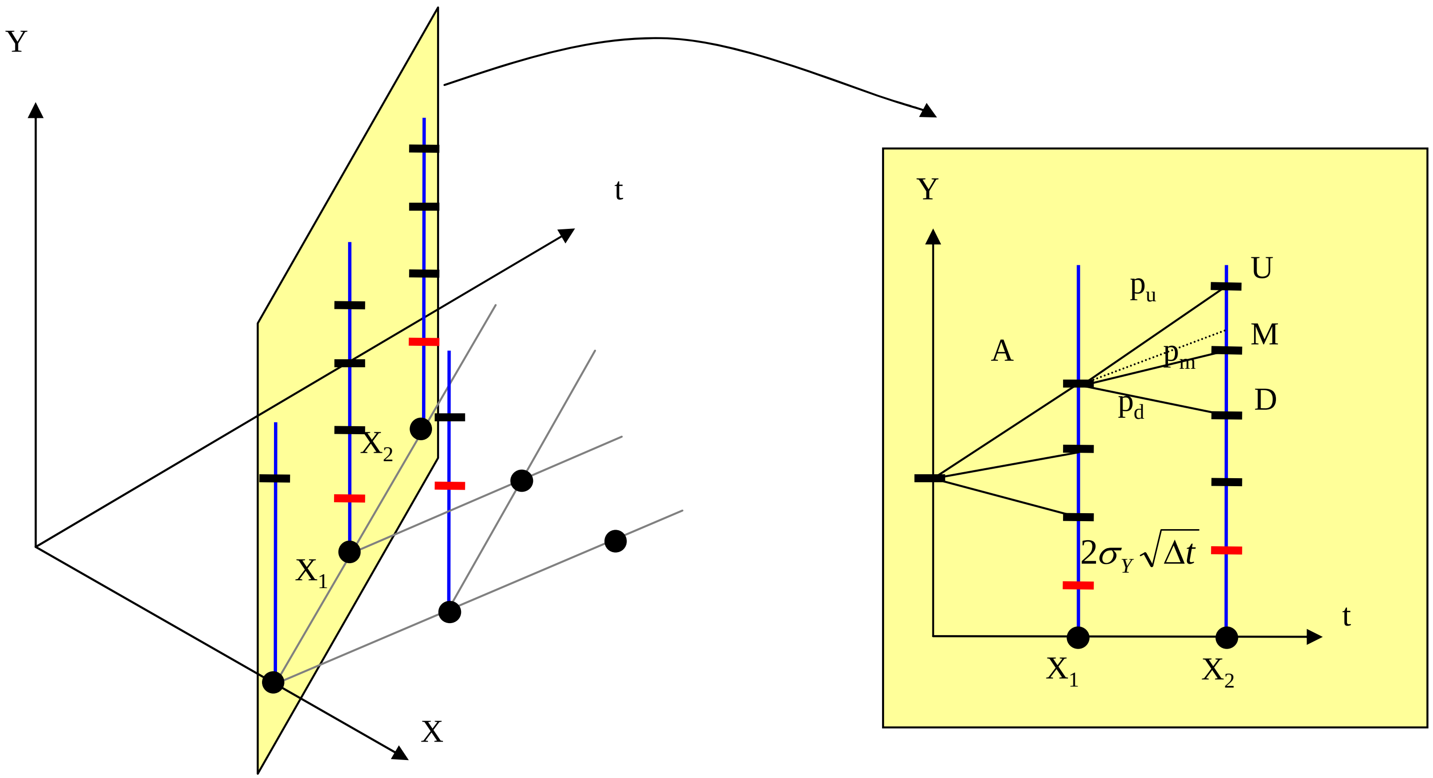


Barrier

- Barrier is $B_V(t) = D^* \cdot e^{-r(T-t)} + c(t)$
  - For each node in *X* tree, we can compute it's option value *c*(*t*).
- Then we see $B_V(t)$ as the firm value boundary so we can translate it to *Y* domain using the equation

$$B_Y(t) = \frac{1}{\sqrt{1-\rho^2}} \left[\frac{1}{\sigma_V} \ln \frac{B_V(t)}{V(0)} - \frac{\rho}{\sigma_S} \ln \frac{S(t)}{S(0)} \right]$$

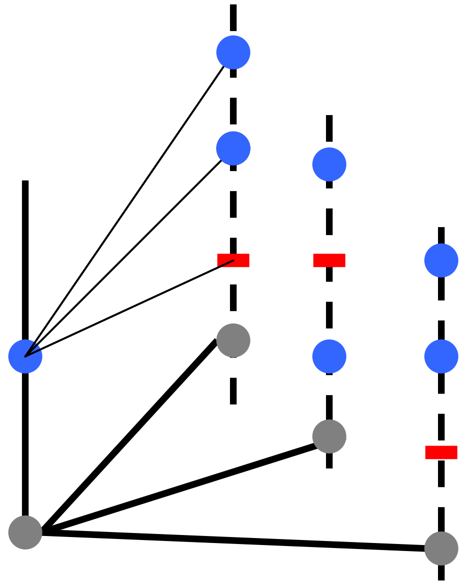
Construct whole tree



- Using the stair tree to construct whole tree.
- And remember the max and min in Y for each X node.

Backward induction

- 9 nodes take expectation to 1 node.
  - Probability simply times X Prob. and Y Prob.
    - X and Y correlation = 0.



Result

Merton	Barrier	Vanilla
D*	潘政宏 (2010)	Klein (1996)
D*+c		Klein (2001)

FPM	Barrier	Vanilla
D*		潘政宏 (2010)
D*+c		

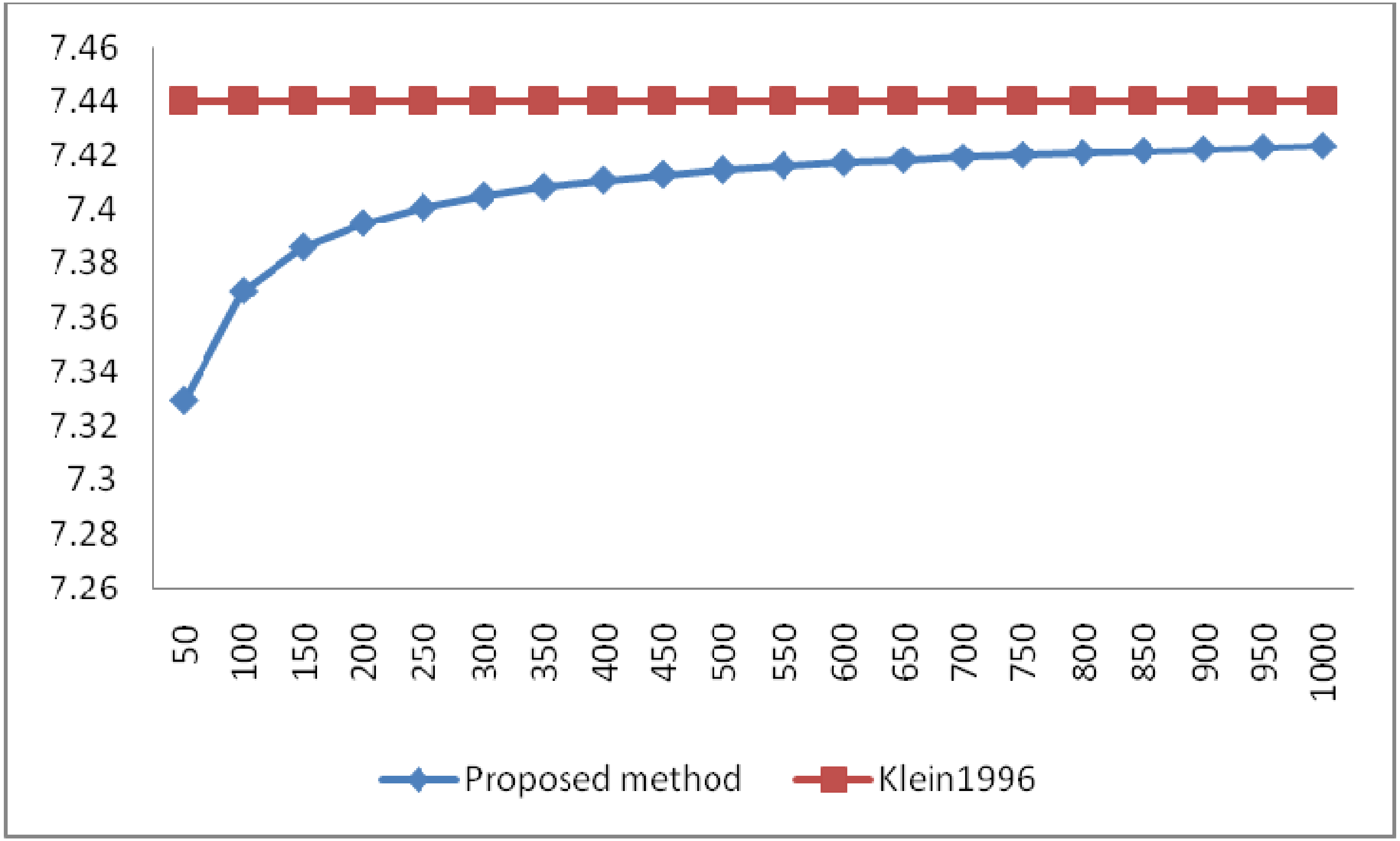
- 我的論文可處理上面八種情況，而其他人目前已做的相關論文如上表。

Result

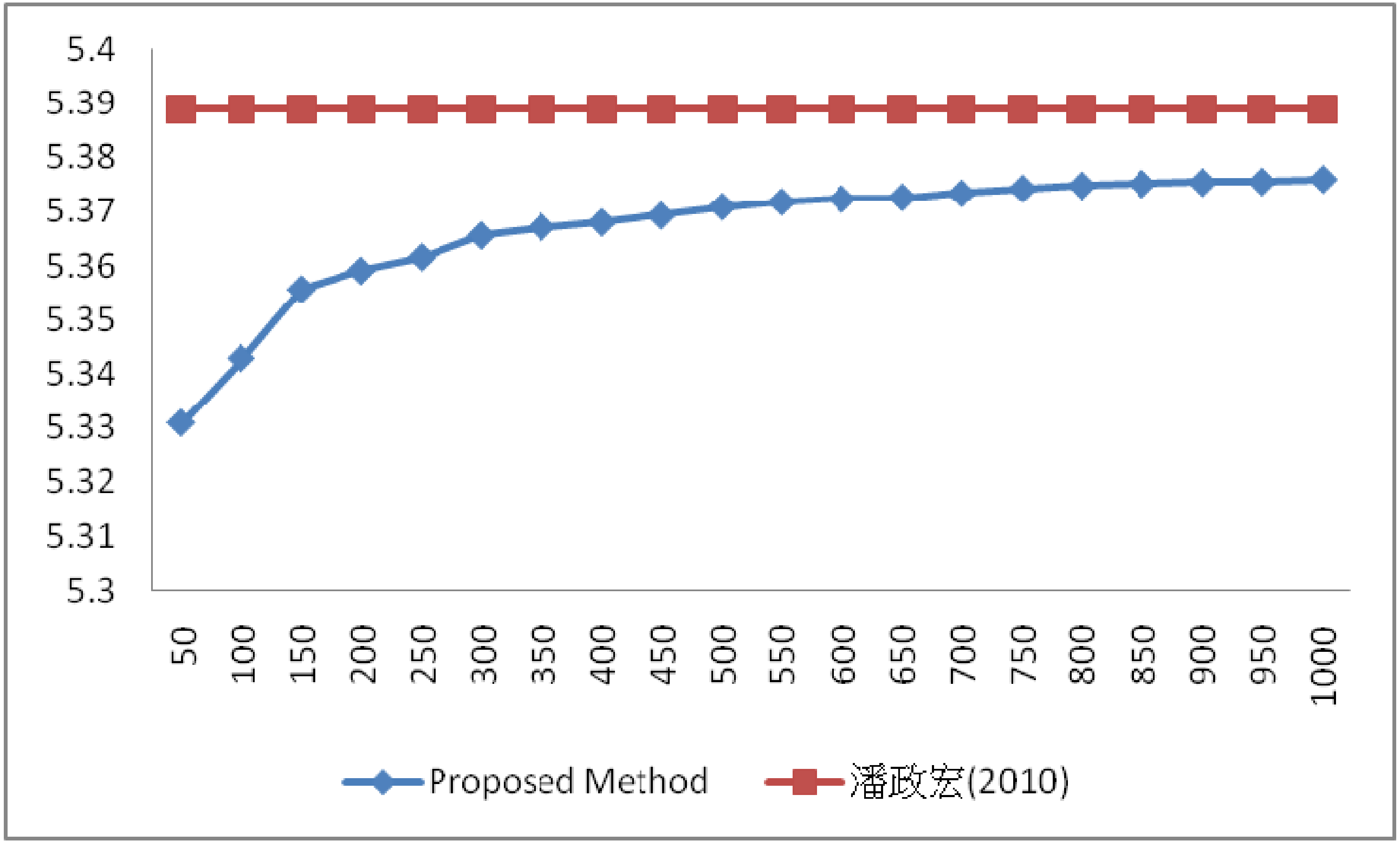
	Base Case	S=30	S=50	V=90	V=110	ρ =0.5	ρ =-0.5	σ_S =0.15	σ_S =0.25	σ_V =0.15	σ_V =0.15
Klein (1996)	7.44	2.27	14.75	7.03	7.74	8.06	6.59	6.45	8.48	7.80	7.10
Proposed method	7.42	2.27	14.71	7.02	7.72	8.05	6.57	6.43	8.46	7.78	7.07
Proposed + FPM	7.28	7.22	14.43	6.88	7.60	7.71	6.85	6.31	8.30	7.60	7.06

	Base Case	S=30	S=50	V=90	V=110	ρ =0.5	ρ =-0.5	σ_S =0.15	σ_S =0.25	σ_V =0.15	σ_V =0.15
Klein (2001)	6.24	2.01	11.59	5.71	6.65	7.36	5.23	5.66	6.70	6.47	5.98
Proposed Method	6.22	2.01	11.58	5.70	6.67	7.34	5.23	5.66	6.71	6.46	5.99
Proposed + FPM	6.63	2.10	12.70	6.38	6.90	7.07	6.41	5.84	7.45	6.70	6.57

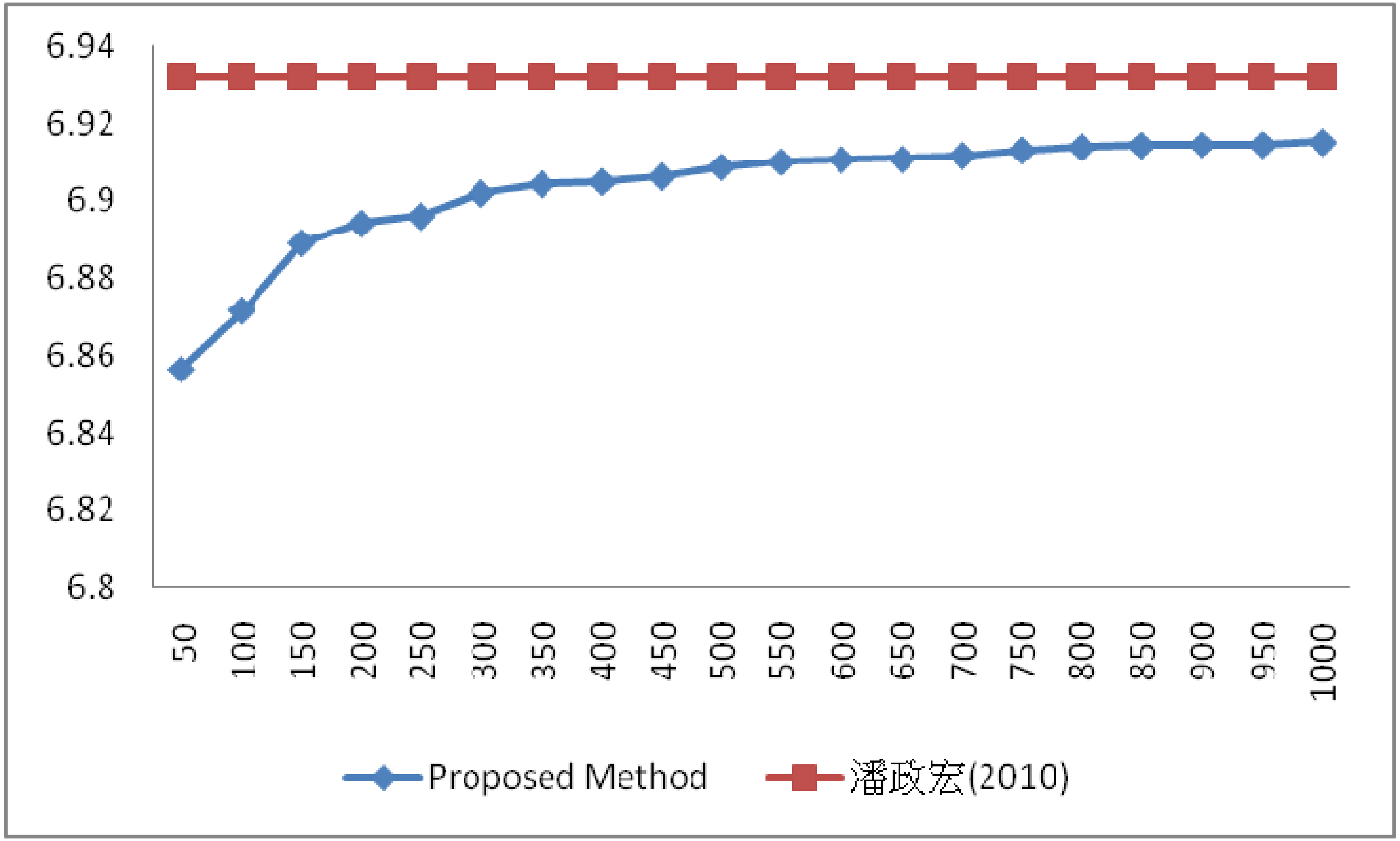
Converge – Vanilla Option



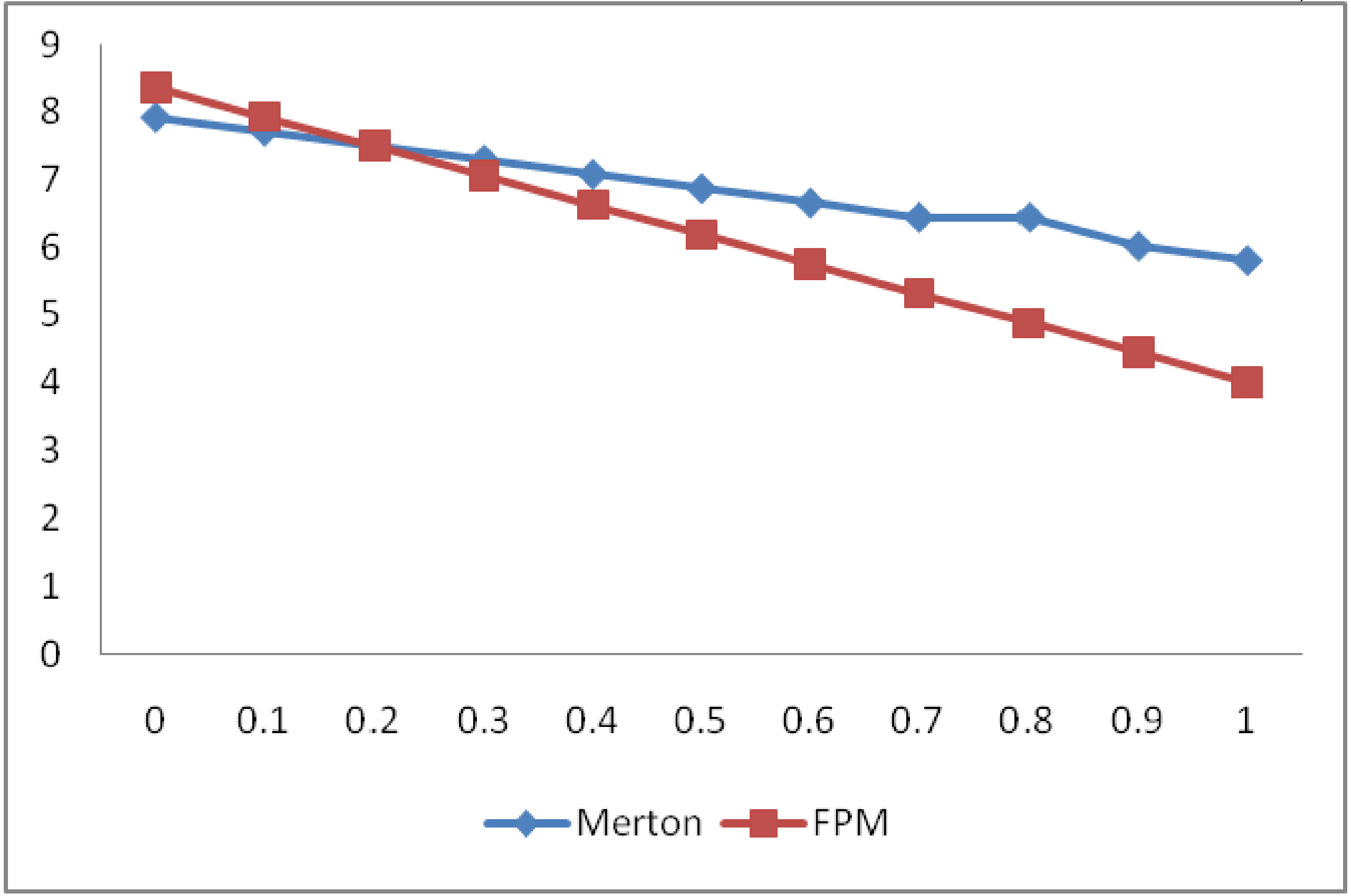
Converge –
Const Barrier Option



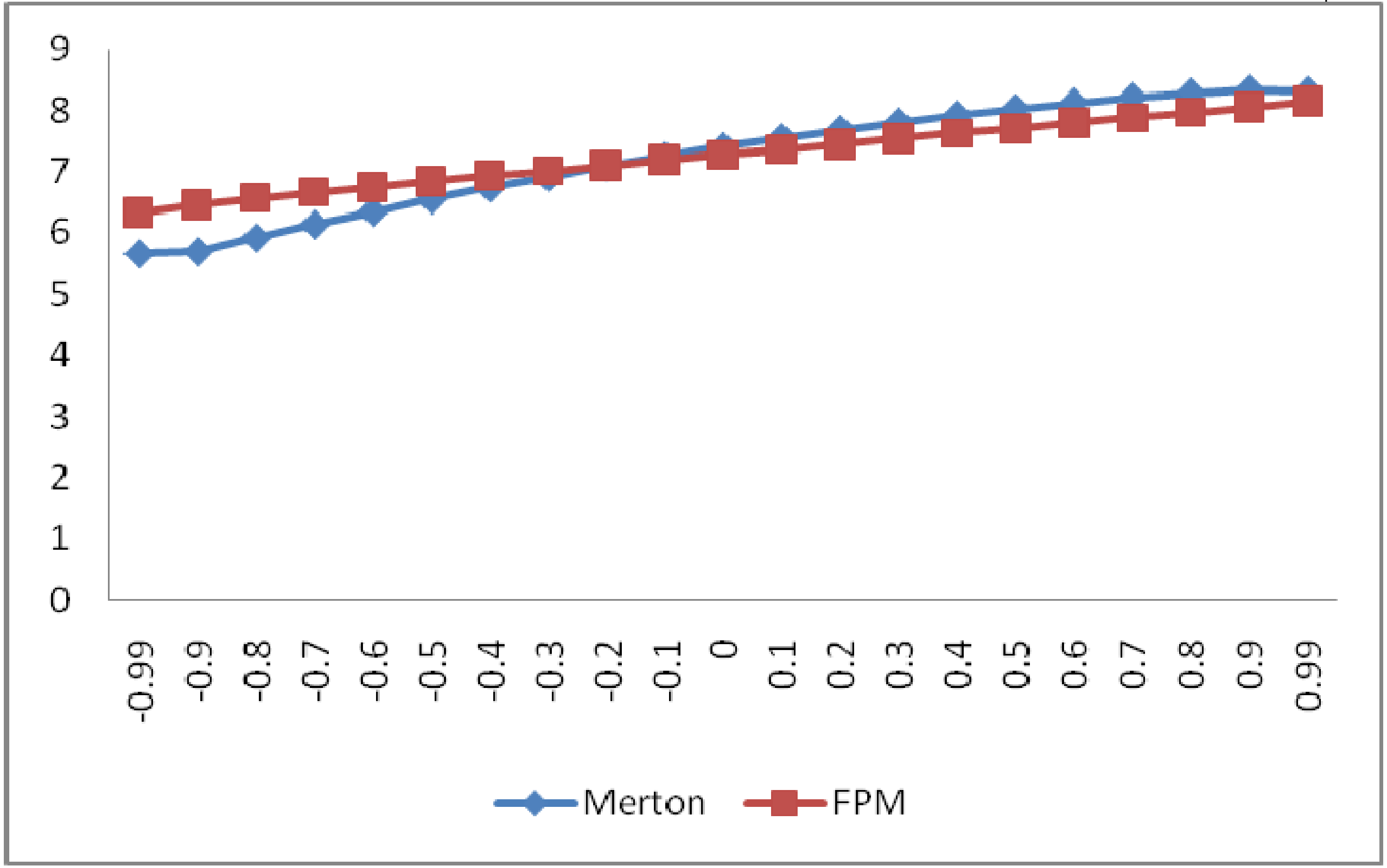
Converge –
Exp Barrier Option



bankruptcy cost (alpha) impact



correlation (rho) impact



Conclusion

- We extend the counterparty default time which can happen before maturity by applying FPM model.
- We can value most exotic options which can be valued by a tree structure with counterparty risk.
- We provide an example that value a barrier option and the result will converge to analytical result.

Thank you!!